APPLICATION OF THE "STRAIGHT-LINES" METHOD TO SOLVING THE ONE-DIMENSIONAL EQUATION OF HEAT CONDUCTION WITH LOCALIZED INHOMOGENEITIES

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The method of product functions is used for obtaining an exact solution to the infinite system of first-order equations which represents an analog of the one-dimensional heat conduction equation with a locally variable thermal conductivity.

It would be of interest to construct explicit solutions to the system of ordinary differential equations which result if one of the differential operators is replaced by its difference analog. With the knowledge of the exact explicit solution to such a system of equations, it will be possible to analyze the trend of the sought function without cumbersome calculations. Finding the numerical values of a function in this way is sometimes not more laborious than calculations based on the exact analytical solution to the equation, if the latter is known. A very simple example is that of solving the Cauchy problem of heat conduction for the two-dimensional case [1-4].

We will construct here the solution to the more elementary problem concerning the infinite system of first-order equations which is an analog of the one-dimensional heat conduction equation with a locally variable thermal conductivity. The original system of equations is [5, 6]

$$\frac{dU_n}{d\tau} = \frac{1}{2}(U_{n-1} - 2U_n + U_{n+1}),\ n \neq k,\ n = 0,\ \pm 1,\ \pm 2,$$
$$g\frac{dU_k}{d\tau} = \frac{1}{2}(U_{k-1} - 2U_k + U_{k+1}). \tag{1}$$

Here U_n is the temperature on a straight-line segment corresponding to the n-th step of a subdivision and g is a coefficient which characterizes the variation of the material properties on the k-th segment.

Letting

$$U_n(\tau) = \varphi_n(\tau)\exp(-\tau) \tag{2}$$

and introducing the product function

$$G(\tau, s) = \sum_{n=-\infty}^{\infty} \varphi_n(\tau) s^n,$$

we go from Eq. (1) to the following expression for $G(\tau, s)$:

$$G(\tau, s) = G(0, s)\exp\frac{1}{2}\left(s + \frac{1}{s}\right)\tau$$
$$+ (1-g)s^k \int_0^\tau (\dot{\varphi}_k - \varphi_k)(v)\exp\frac{1}{2}\left(s + \frac{1}{s}\right)(\tau - v)\,dv. \tag{3}$$

Expanding (3) into powers in s, we obtain

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$$\varphi_n(\tau) = U_m(0)\, I_{n-m}(\tau) + (1-g)\int_0^{\tau} (\dot{\varphi}_k - \varphi_k)(v)\, I_{n-k}(\tau - v)\, dv. \tag{4}$$

Nonzero initial conditions are assumed to apply only to the m-th segment. Thus the resulting solution will represent the function of a point source. The solution for an arbitrary initial temperature distribution is then found by summation over m.

If n = k, then relation (4) becomes an integral equation for function $\varphi_k(\tau)$:

$$\varphi_k(\tau) = U_m(0)\, I_{k-m}(\tau) + (1-g)\int_0^{\tau} (\dot{\varphi}_k - \varphi_k)(v)\, I_0(\tau - v)\, dv. \tag{5}$$

If the character of φ_k as a function of time is determined from Eq. (5), then inserting $\varphi_k(\tau)$ into (4) will yield analogous time functions for any segment of the infinite straight line. The basic difficulty, then, lies in solving Eq. (5). A Laplace transformation with respect to τ and a few elementary operators will yield

$$\varphi_k(\tau) = \frac{U_m(0)}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(p\tau)\, dp}{\{\sqrt{p^2-1} - (p-1)(1-g)\}\,(p+\sqrt{p^2-1})^{|k-m|}}. \tag{6}$$

The contour integral in (6) is conveniently calculated after a change of variables

$$w = p - \sqrt{p^2-1} \tag{7}$$

has been made.

Function (7) maps conformally the plane of variable p cut along the real axis from –1 to + 1 onto a unit circle in the w-plane, while tracing the contour clockwise in the p-plane corresponds to tracing it counter-clockwise in the w-plane. We have

$$\varphi_k(\tau) = -\frac{U_m(0)}{2\pi i\,(g-2)}\int_l \frac{(w+1)w^{|k-m|-1}\exp\frac{1}{2}\left(w+\frac{1}{w}\right)\tau}{w - \frac{g}{g-2}}\, dw. \tag{8}$$

When g > 1, the zero of the denominator in (8) is located outside the unit circle. Therefore, one may integrate (8) along any contour inside the unit circle encompassing the point w = 0, where as essential singularity exists. Expanding the integrand function in (8) into a Laurent series, we deduce

$$\varphi_k(\tau) = U_m(0)\sum_{r=0}^{\infty} \frac{(g-2)^r}{g^{r+1}}\,[I_{|k-m|+r} + I_{|k-m|+r+1}](\tau). \tag{9}$$

Although expression (8) is meaningless for g = 2, expression (9) is valid for that case too. The simplest special cases are:

$$\varphi_k(\tau) = U_m(0)\, I_{k-m}(\tau), \quad g = 1; \tag{10}$$

$$\varphi_k(\tau) = \frac{1}{2}\, U_m(0)\,[I_{k-m} + I_{|k-m|+1}](\tau), \quad g = 2. \tag{11}$$

Formulas (6), (8)-(11) are based on the assumption that m ≠ k. If m = k, however, then the factor g must be added in formulas (9)-(11).

Inserting (9) into (4) yields the general expression for function $\varphi_n(\tau)$:

$$\varphi_n(\tau) = U_m(0)\Big\{I_{n-m} + \frac{1-g}{g}\, I_{|k-m|+|n-k|} + 2(g-1)\sum_{r=0}^{\infty} \frac{(g-2)^r}{g^{r+2}}\, I_{|k-m|+|n-k|+r+1}\Big\}(\tau). \tag{12}$$

It is easy to see that expression (12) embraces (9)-(11). The practical application of (12) is not difficult, since it is made up entirely of tabulated functions with very simple coefficients. It can also be readily rewritten in terms of Lommel functions in two variables [7].

The preceding analysis was applied to the Cauchy problem on an infinite straight line. We will now show how a solution can be found for the case of a semiinfinite medium. Let the segment where n = 0 be at the boundary. Then, if an existing semiinfinite straight-line segment is connected to its mirror image behind the boundary, the problem considered together with the inhomogeneity and the initial distribution reduces again to the problem of an infinite straight line. A positive fictitious source (even continuation) will correspond to the problem with thermal insulation at the surface, while a negative one (odd continuation) will correspond to the problem with zero boundary values. It is required, then, to find the solution to the system of equations

$$\begin{aligned} \frac{dU_n}{d\tau} &= \frac{1}{2}(U_{n-1} - 2U_n + U_{n+1}), \quad n \neq k, \ -k-1, \\ g\frac{dU_n}{d\tau} &= \frac{1}{2}(U_{n-1} - 2U_n + U_{n+1}), \quad n = k, \ -k-1 \end{aligned} \tag{13}$$

with the initial distribution

$$U_n(0) = 0, \ n \neq m, \ -m-1, \ U_m(0) > 0, \ U_{-m-1}(0) = \pm U_m(0).$$

For an arbitrary function, the equation will be

$$\frac{dG}{d\tau} = \frac{1}{2}\left(s + \frac{1}{s}\right)G + (1-g)\left[(\dot{\varphi}_k - \varphi_k)s^k + (\dot{\varphi}_{-k-1} - \varphi_{-k-1})s^{-k-1}\right].$$

Considering that in the problem discussed here the initial conditions are given as $\varphi_{-k-1}(\tau) = \pm\varphi_k(\tau)$, we find

$$\varphi_n(\tau) = U_m(0)(I_{n-m} \pm I_{n+m+1})(\tau) + (1-g) \times \int_0^\tau (\dot{\varphi}_k - \varphi_k)(v)(I_{n-k} \pm I_{n+k+1})(\tau - v)\,dv.$$

All subsequent operations are analogous to these. If the medium is confined on both sides, however, then a general solution to the problem can also be found by the method of [8], for example.

NOTATION

- U is the temperature;
- n is the segment number of the subdivision;
- k is the number of irregular segment;
- τ is the dimensionless time;
- φ_n is the auxiliary function;
- G is the product function;
- s is the parameter;
- v is the integration variable;
- I_i is the Bessel function of i-th order, of an imaginary argument;
- m is the number of segment with prescribed initial temperature;
- p, w are the complex variables.

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